

DECOMPOSING MANIFOLDS INTO HOMOLOGICALLY EQUIVALENT SUBMANIFOLDS⁽¹⁾

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1. Introduction. The basic question asked in this paper is: Under what conditions does a closed, combinatorial manifold contain a compact submanifold whose homology groups are isomorphic to those of the closure of its complement?

If R is a principal ideal domain, two topological spaces X and Y are said to be R -equivalent if their graded singular homology modules $H_*(X; R)$ and $H_*(Y; R)$ with coefficients in R are R -isomorphic. If X and Y are \mathbb{Z} -equivalent, where \mathbb{Z} is the ring of integers, we say that X and Y are homologically equivalent. By universal coefficients, homologically equivalent spaces are R -equivalent for any principal ideal domain R .

With this terminology the main results can be stated. Theorem 1 is almost immediate, and most of the paper is devoted to proving Theorems 2 and 4 which are partial converses. Theorem 4 is a corollary to a slightly stronger result (Theorem 3) but the simpler statement is given here.

THEOREM 1. *If a closed manifold M contains a compact submanifold N which is R -equivalent to $\text{cl}(M - N)$ for some principal ideal domain R , then the Euler characteristic of M is even.*

THEOREM 2. *Suppose M is a closed, combinatorial manifold of odd dimension $2m + 1$, and that N is a regular neighborhood of the m -skeleton of some combinatorial triangulation of M . Then N is homologically equivalent to $\text{cl}(M - N)$.*

THEOREM 4. *Suppose M is a closed, combinatorial manifold of even dimension and that the Euler characteristic of M is even. Then M contains a PL submanifold N which is \mathbb{Z}_2 -equivalent to $\text{cl}(M - N)$.*

2. Terminology and definitions. With the exception of Theorem 1, whose proof is valid for any manifold, all spaces and maps in this paper are in the polyhedral category in the sense of [3] or [7]. Thus all manifolds will have a combinatorial triangulation, and all subspaces and maps will be piecewise linear (PL). The theory of regular neighborhoods will be assumed.

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Let M be a closed (i.e. compact and without boundary) n -manifold. If A and B are simplices in a combinatorial triangulation K of M , $b(A)$ will denote the barycenter of A , $A < B$ means A is a face of B , and K' and K'' refer to the first and second barycentric subdivisions of K . If L is a subcomplex of K , the underlying topological space is denoted by $|L|$. The dual cell A^* of a simplex $A \in K$ is the union of all simplices of K' of the form $|\{b(A_j), \dots, b(A_k)\}|$, where $A < A_j \cdots < A_k$. Then the dual cell decomposition of M is defined as $K^* = \{A^* : A \in K\}$. For a subcomplex $L \subset K$, we define the complementary cell complex $L_* \subset K^*$ by

$$L_* = \{A^* \in K^* : A \in K - L\}.$$

We recall some basic facts about the dual cell decomposition. K' is a simplicial subdivision of both K and K^* . Both K and K^* are spherical complexes in the sense of [2, §19], so that the homology modules of M in dimensions less than r can be computed from either r -skeleton K^r or K^{*r} . If A is a k -simplex in K , then $A^* \in K^*$ is a cell of dimension $n - k$. Thus if L is the r -skeleton of K , L_* is the $(n - r - 1)$ -skeleton of K^* . The following fact is used in Zeeman's presentation of Stallings' proof of the Poincaré conjecture ([6, p. 200] or [7, Chapter VII, p. 8]), and is stated here for easy reference.

LEMMA 0. *Let L be a proper subcomplex of a combinatorial triangulation K of a closed manifold M , and let $L_* \subset K^*$ be the complementary subcomplex of the dual cell decomposition. Then if N and N_* are the simplicial neighborhoods respectively of $|L|$ and $|L_*|$ in K'' , $N_* = \text{cl}(M - N)$.*

Throughout the discussion R will denote an arbitrary principal ideal domain, \mathbb{Z} the ring of integers, and \mathbb{Z}_2 the integers modulo 2. The singular homology module $H_i(X; R)$ of a space X will always be considered as an R -module with $F_i(X; R)$ and $T_i(X; R)$ representing the torsion-free and torsion submodules respectively. Recall [2, p. 136] that if the homology of X is finitely generated there is an isomorphism $H^i(X; R) \cong F_i(X; R) \oplus T_{i-1}(X; R)$. By $\text{rank } H_i(X; R)$ we mean the rank of $H_i(X; R)$ as an R -module. Then the Euler characteristic of X is given by

$$\chi(X) = \sum (-1)^i \text{rank } H_i(X; R),$$

which by universal coefficients is independent of R .

3. Proofs of Theorems 1 and 2.

Proof of Theorem 1. Suppose the dimension of M is n and denote $N_* = \text{cl}(M - N)$. If n is odd, $\chi(M) = 0$ by Poincaré duality, so we consider the case that n is even. From the Mayer-Vietoris sequence (coefficients in R) of the couple $\{N, N_*\}$ and the fact that in an exact sequence of R -modules with 0 at each end the alternating sum of the ranks is 0 (see [2, p. 100]), it follows that

$$\chi(M) = \chi(N) + \chi(N_*) - \chi(N \cap N_*).$$

But $N \cap N_* = \text{bd } N$ is a manifold of odd dimension, and $\chi(N) = \chi(N_*)$ since N and N_* are R -equivalent. Thus $\chi(M) = 2\chi(N)$.

REMARK. The above proof depends only on the ranks of the homology modules of N and N_* . Thus the conclusion of Theorem 1 holds if M contains a submanifold N such that $\text{rank } H_i(N; R) = \text{rank } H_i(N_*; R)$ for each i , where $N_* = \text{cl}(M - N)$.

Proof of Theorem 2. Suppose that K is a combinatorial triangulation of M and that L is the m -skeleton of K . If N is a regular neighborhood of $|L|$, we must show that N is Z -equivalent to $N_* = \text{cl}(M - N)$.

By the uniqueness of regular neighborhoods [7, Chapter III], we can assume that N is the simplicial neighborhood of $|L|$ in K'' . Then by Lemma 0, N_* is the simplicial neighborhood of the complementary cell complex $|L_*|$ in K'' . Since L_* is the m -skeleton of the dual cell decomposition K^* of M , it follows that for any R ,

$$(1) \quad \begin{aligned} H_i(N; R) &\cong H_i(M; R) \cong H_i(N_*; R) & \text{if } i < m, \\ H_i(N; R) &\cong 0 \cong H_i(N_*; R) & \text{if } i > m. \end{aligned}$$

Then to show that N and N_* are homologically equivalent, it remains to check the modules in dimension m .

We first consider the case $R = Z_2$. Since all manifolds are orientable over Z_2 , there is by Lefschetz duality a sign commutative diagram (coefficients in Z_2)

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^m(N, \text{bd } N) & \longrightarrow & H^m(N) & \longrightarrow & H^m(\text{bd } N) & \longrightarrow & H^{m+1}(N, \text{bd } N) & \longrightarrow & H^{m+1}(N) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_{m+1}(N) & \longrightarrow & H_{m+1}(N, \text{bd } N) & \longrightarrow & H_m(\text{bd } N) & \longrightarrow & H_m(N) & \longrightarrow & H_m(N, \text{bd } N) & \longrightarrow & \cdots \end{array}$$

with the rows exact and the vertical arrows isomorphisms. Because L is an m -dimensional complex, $H^{m+1}(N) \cong 0 \cong H_{m+1}(N)$. Since all Z_2 -modules are free, $H^m(N) \cong H_m(N)$, and it follows that $H_m(\text{bd } N) \cong H_m(N) \oplus H_m(N)$. An identical argument shows that $H_m(\text{bd } N_*) \cong H_m(N_*) \oplus H_m(N_*)$. But $\text{bd } N = \text{bd } N_*$, implying that $H_m(N; Z_2) \cong H_m(N_*; Z_2)$. Thus N is Z_2 -equivalent to N_* and in particular $\chi(N) = \chi(N_*)$.

Now calculating the Euler characteristics of N and N_* from the homology with coefficients in Z , it follows from equations (1) above that $\text{rank } H_m(N; Z) = \text{rank } H_m(N_*; Z)$. But since L and L_* are m -dimensional, these two Z -modules are free and hence isomorphic, completing the proof.

4. The even dimensional case. In this section the following theorem is proved:

THEOREM 3. Suppose M is a closed, combinatorial manifold of even dimension $2m \geq 4$, that M is orientable over the principal ideal domain R , and that the Euler characteristic of M is even. Then M contains a compact PL submanifold N satisfying:

$$H_j(N) \cong H_j(M) \cong H_j(N_*) \text{ for } j < m-1,$$

$$\text{rank } H_{m-1}(N) = \text{rank } H_{m-1}(M) = \text{rank } H_{m-1}(N_*),$$

$$H_m(N) \cong H_m(N_*) \cong \text{a free } R\text{-module with rank} = \frac{1}{2} \text{rank } H_m(M),$$

$$H_j(N) = 0 = H_j(N_*) \text{ for } j > m,$$

where $N_* = \text{cl}(M - N)$ and coefficients are in R .

Notice that such an N is almost R -equivalent to N_* ; the only problem is that $H_{m-1}(N)$ and $H_{m-1}(N_*)$ may have nonisomorphic torsion submodules. However this problem is avoided if R is a field, and in this case N and N_* are R -equivalent. In particular, since every manifold is orientable over Z_2 , Theorem 4 is an immediate corollary.

The exact proof of Theorem 2 does not work here since a triangulation K of M does not have complementary skeletons of the same dimension. However the approach is the same in that we attempt to find complementary subcomplexes of K and the dual cell decomposition K^* which have isomorphic homology modules. The general outline of the proof of Theorem 3 is as follows: Let L_0 be the $(m-1)$ -skeleton of K so that L_{0*} is the m -skeleton of K^* . Then the homology modules of L_0 and L_{0*} are isomorphic to those of M in dimensions less than $m-1$ and are 0 for dimensions greater than m . We then show that it is possible to adjoin m -simplices of K to L_0 and delete the corresponding dual simplices from L_{0*} to form L and L_* which are almost R -equivalent. The details are given below.

Suppose M has a cell decomposition with m -cells C_1, \dots, C_t . (In the proof these will be either the m -simplices of K or their dual cells in K^* .) Let L_0 be the $(m-1)$ -skeleton of the cell decomposition, and let $L_i = L_{i-1} \cup \{C_i\}$, $i = 1, \dots, t$. Thus L_t is the m -skeleton, and $|L_i|$ is obtained from $|L_{i-1}|$ by attaching the m -cell C_i via the identity map on its boundary. The following facts are known (see [2, pp. 89, 101]; all coefficients are in R):

- A. $H_m(L_i)$ is a free R -module, $i = 0, 1, \dots, t$.
- B. $H_j(L_i) \cong H_j(M)$ for $j < m-1$, $i = 0, 1, \dots, t$.
- C. $H_j(L_i) = 0$ for $j > m$, $i = 0, 1, \dots, t$.
- D. For $i = 0, \dots, t-1$, precisely one of the following holds:
 - 1. $H_m(L_{i+1}) \cong H_m(L_i)$ and $\text{rank } H_{m-1}(L_{i+1}) = \text{rank } H_{m-1}(L_i) - 1$.
 - 2. $H_m(L_{i+1}) \cong H_m(L_i) \oplus R$ and $\text{rank } H_{m-1}(L_{i+1}) = \text{rank } H_{m-1}(L_i)$.

LEMMA 1. *With the above notation the m -cells of the cell decomposition can be ordered so that the following hold:*

- E. $H_m(L_i) = 0$ for $i = 0, \dots, r$.
- F. $H_m(L_i) = H_m(L_{i-1}) \oplus R$ for $i = r+1, \dots, t$.
- G. $\text{rank } H_{m-1}(L_i) = \text{rank } H_{m-1}(M)$ for $i = r, \dots, t$.

Moreover, having chosen C_1, \dots, C_r , the ordering of C_{r+1}, \dots, C_t is arbitrary.

Proof. Let $S = \{C_1, \dots, C_r\}$ be a maximal set of m -cells having the property that $H_m(L_r) = 0$; that is, for any additional m -cell C , $H_m(L_r \cup C) \cong R$. Order the remaining m -cells arbitrarily.

Now suppose for some $i > r$, $H_m(L_i) \cong H_m(L_{i-1})$. By [2, p. 89], $\text{kernel } H_{m-1}(f) = 0$, where $f: \text{bd } C_i \rightarrow |L_{i-1}|$ is the identity map and $H_{m-1}(f)$ is the induced homomorphism on homology. But f factors as

$$\text{bd } C_i \xrightarrow{f'} |L_r| \xrightarrow{j} |L_{i-1}|,$$

where f' is also the identity and j is inclusion. Since $H_{m-1}(f) = H_{m-1}(j)H_{m-1}(f')$, kernel $H_{m-1}(f') = 0$. But then $H_m(L_r \cup C_i) = 0$, contradicting the maximality of S .

It now follows from D above that E, F, and G are satisfied.

LEMMA 2. *Let K be a combinatorial triangulation of the $2m$ -manifold M which is orientable over R . Suppose $L \subset K$ is a subcomplex of dimension $\leq m$ satisfying $H_m(L; R) = 0$. Then*

$$\text{rank } H_{m-1}(L_*; R) = \text{rank } H_{m-1}(M; R).$$

Proof. Let N and N_* be the simplicial neighborhoods of $|L|$ and $|L_*|$ respectively in K'' as in Lemma 0. By excision and Lefschetz duality

$$\begin{aligned} H_m(M, N_*) &\cong H_m(N, \text{bd } N) \cong H^m(N) \cong F_m(N) \oplus T_{m-1}(N) \\ &\cong F_m(L) \oplus T_{m-1}(L) \cong T_{m-1}(L). \end{aligned}$$

Similarly, $H_{m-1}(M, N_*) \cong F_{m+1}(L) \oplus T_m(L) = 0$. Then in the long exact sequence of the pair (M, N_*) we have

$$\cdots \rightarrow T_{m-1}(L) \rightarrow H_{m-1}(N_*) \rightarrow H_{m-1}(M) \rightarrow 0 \rightarrow \cdots$$

It follows that the free submodules of $H_{m-1}(N_*)$ and $H_{m-1}(M)$ are isomorphic, completing the lemma.

Proof of Theorem 3. Let K be a combinatorial triangulation of M and K^* the dual cell decomposition. Denote by L_0 the $(m-1)$ -skeleton of K and by C_1, \dots, C_t the m -simplices in K . Using the notation established above and in Lemma 1, we can order these m -simplices so that conditions A through G hold.

Note that for $i = 1, \dots, t$, $|L_{i-1*}|$ is obtained from $|L_{i*}|$ by attaching the dual cell C_{i*} . In particular, L_{r*} is obtained from L_{i*} (which is $(m-1)$ -skeleton of K^*) by attaching $C_{i*}, C_{i-1*}, \dots, C_{r+1*}$ in that order. Applying Lemma 1 to $|L_{r*}|$, these dual cells can be ordered so that

$$\text{E}^*. \quad H_m(L_{i*}) = 0 \text{ for } s \leq i \leq t,$$

$$\text{F}^*. \quad H_m(L_{i*}) = H_m(L_{i+1*}) \oplus R \text{ for } r \leq i < s,$$

and conditions A through G still hold for this ordering. Note also that analogous conditions A*, B*, and C* hold for each L_{i*} . By Lemma 2, $\text{rank } H_{m-1}(L_{r*}) = \text{rank } H_{m-1}(M)$, and from F*, D, and G the following condition holds:

$$\text{H}. \quad \text{rank } H_{m-1}(L_{i*}) = \text{rank } H_{m-1}(M) = \text{rank } H_{m-1}(L_i), \text{ for } r \leq i \leq s.$$

For $i = r, r+1, \dots, s$, let N_i and N_{i*} be the simplicial neighborhoods in K'' of $|L_i|$ and $|L_{i*}|$ respectively, as in Lemma 0. Since $N_i \cap N_{i*} = \text{bd } N_i$ is an odd dimensional manifold,

$$\chi(M) = \chi(N_i) + \chi(N_{i*}) = \chi(L_i) + \chi(L_{i*}).$$

Consider in particular the case $i = s$. From B, H, and E* it follows that

$$\chi(M) = 2 \sum_{q=0}^{m-1} (-1)^q \text{rank } H_q(M) + (-1)^m \text{rank } H_m(L_s).$$

But then from Poincaré duality and the hypothesis that $\chi(M)$ is even we find that $\text{rank } H_m(L_s) = \text{rank } H_m(M)$ and that this is an even number. Now because F holds, $\text{rank } H_m(L_s) = s - r$ and $r + s$ is even. Let $p = (r + s)/2$. Then from F and F^* ,

$$\text{rank } H_m(L_p) = (s - r)/2 = \text{rank } H_m(L_{p*}),$$

and because these R -modules are free, $H_m(L_p) \cong H_m(L_{p*})$. Letting $N = N_p$, $N_* = N_{p*}$ completes the proof of Theorem 3.

For even dimensional manifolds one would hope for a result better than Theorem 3. Such a result would give a \mathbb{Z} -decomposition for any $2m$ -manifold, or at least for one which is orientable over \mathbb{Z} . I have not been able to improve the technique used in Theorem 3 to eliminate the possibility of different torsion submodules in dimension $m - 1$. Whether such an improvement is possible or a different approach will produce homologically equivalent submanifolds remains an open question. However, with the additional hypothesis that $H_m(M; R) = 0$ a stronger conclusion is obtained.

THEOREM 5. *Let M be as in Theorem 3 with the additional hypothesis that $H_m(M; R) = 0$. Then the submanifold N obtained in Theorem 3 is R -equivalent to $N_* = \text{cl}(M - N)$.*

Proof. We need only show that $T_{m-1}(N; R) \cong T_{m-1}(N_*; R)$. By duality, $H^m(M) = 0$, so from the cohomology sequence of the pair (M, N) there is a monomorphism $H^m(N) \subset H^{m+1}(M, N)$. Since $F_m(N) = 0$, $H^m(N) \cong T_{m-1}(N)$. By excision and Lefschetz duality, $H^{m+1}(M, N) \cong H^{m+1}(N_*, \text{bd } N_*) \cong H_{m-1}(N_*)$. Thus there is a monomorphism $T_{m-1}(N) \subset T_{m-1}(N_*)$. Reversing the roles of N and N_* gives $T_{m-1}(N) \cong T_{m-1}(N_*)$ since the modules are finite.

5. Two remarks.

1. *Uniqueness of the decompositions.* It should be noted that the R -equivalent decompositions given in the theorems above are not unique. It is not difficult to produce examples of a manifold having different submanifolds each homologically equivalent to its complement but not equivalent to each other. It is true that the decomposition produced in Theorem 4 is unique in the sense that the homology modules of the submanifold N are determined by those of M . This is not true of the submanifold produced in Theorem 2. In this case the homology of N in dimension $(n - 1)/2$ depends on the triangulation K and may be totally unrelated to that of M . However, if the $(2m + 1)$ -manifold M is orientable over a field F it is possible, using the techniques of Theorem 3, to find a submanifold N which is F -equivalent to its complement and such that $H_i(N; F)$ is isomorphic to $H_i(M; F)$ for $i \leq m$ and is 0 for $i > m$. A discussion of these aspects of uniqueness is found in [1].

2. *Dimensions 2 and 3.* If M is a closed 2-manifold with even Euler characteristic, then from the classification theorem [4, Chapter 1] M can be represented as the connected sum of tori if it is orientable or as the connected sum of an even number of projective planes if it is nonorientable. In either case it is easy to see that M contains a submanifold N such that M is the double of N .

If M is a closed 3-manifold then the submanifolds N and $\text{cl}(M - N)$ of Theorem 2 are known to be handlebodies [5, p. 219] which can be shown to be homeomorphic. Thus the main theorems of this paper can be thought of as generalizations of these facts.

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